Published by Institute of Physics Publishing for SISSA/ISAS
Received: *April 5, 2004*
Revised: *July 8, 2004*
Accepted: *July 15, 2004*

A monopole solution from noncommutative multi-instantons

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Abstract: We extend the relation between instanton and monopole solutions of the selfduality equations in SU(2) gauge theory to noncommutative space-times. Using this approach and starting from a noncommutative multi-instanton solution we construct a U(2) monopole configuration which lives in 3 dimensional ordinary space. This configuration resembles the Wu-Yang monopole and satisfies the selfduality (Bogomol'nyi) equations for a U(2) Yang-Mills-Higgs system.

Keywords: Solitons Monopoles and Instantons, Gauge Symmetry, Non-Commutative Geometry.

[*] CONICET.
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JHEP07(2004)037

Contents

1. Introduction

Soliton and instanton solutions in noncommutative field theories have been the object of many investigations in recent years (see [1, 2] for a complete list of references). Whenever the space-time dimension is even, a connection between the noncommutative algebra and that of creation and annihilation operators in Fock space can be exploited in order to find explicit exact solutions which are the natural extensions of those already constructed in ordinary space. In particular, after the pioneering work of Nekrasov and Schwarz [3] on noncommutative instantons, different approaches have been followed to construct and analyze explicit selfdual solutions [4]–[16].

Concerning solitons, and apart from vortices (see [17] for a complete list of references on this issue), noncommutative monopole configurations have been extensively discussed [18]–[26]. In particular, BPS monopole solutions have been constructed in [24] for noncommutative U(1) and U(2) gauge theories by solving the noncommutative extension of the so-called Nahm equations. An interesting correspondence between the noncommutative monopole solution and a D1 string stretched between D3 branes was revealed by this work.

As it is well known, conventional instanton and monopole solutions can be related. Geometrically, the idea is that if one looks for solutions of the selfduality equations with a U(1) isometry k^μ, then a monopole configuration of a Yang-Mills-Higgs system can be obtained with $\Phi = k^\mu A_\mu$ playing the role of the Higgs scalar in the adjoint. When the isometry is chosen to be along the euclidean time ($\Phi = A_0$) the selfduality equations become the Bogomol'nyi equations for a Yang-Mills-Higgs system in the Prasad-Sommerfield limit. This procedure, originally developed in Refs. [35], starting from an axially symmetric multi-instanton solution with charge q [27], was afterwards extended by Nahm [29] to the ADHM multi-instanton solution. A different choice for k^μ leading to hyperbolic monopoles was originally proposed by Atiyah [30].

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The extension of Nahm's construction to the noncommutative case has been developed in [22]–[24], based on the noncommutative version of the ADHM construction developed in [3]. In the U(1) case, which was studied in detail in [22], a soliton solution having zero magnetic charge was constructed. It can be interpreted as consisting of a monopole attached to a string that runs off to infinity. In order to see whether truly magnetically charged isolated configurations in 3-dimensional noncommutative space can be obtained from 4-dimensional noncommutative instantons we shall extend in this work Manton's proposal of considering the infinite charge limit ($q \to \infty$) of Witten multi-instanton solution. We will use the noncommutative version of Witten's solution constructed in [36], which we review in section 2. Then, in section 3 we discuss the choice of the appropriate gauge condition and discover, as a byproduct, a very peculiar situation that can arise for constant field strengths in noncommutative gauge theories. Indeed, we show that under certain conditions, there exist gauge orbits consisting of just one point. The $q \to \infty$ limit leading to a monopole configuration is considered in section 4 where we write the BPS equations obeyed by the soliton solution. We discuss the properties of the solution, relating it with that of a Dirac monopole. Finally, in section 5 we summarize and discuss our results.

2. The instanton solution

We here briefly review the extension of Witten's multi-instanton solution to noncommutative space, as presented in ref. [36].

The clue in Witten's ansatz [27] is to reduce the four dimensional problem to a two dimensional one through an axially symmetric multi-instanton ansatz. That is, one passes from 4 dimensional euclidean space-time with coordinates $(r, \vartheta, \varphi, t)$ to 2 dimensional curved space-time with coordinates (r, t).

The noncommutative solution in [36] corresponds to a space-time with commutation relations given by

$$\begin{aligned} [r, t] &= i\theta(r, t) \\ [r, \vartheta] &= [r, \varphi] = [t, \vartheta] = [t, \varphi] = [\vartheta, \varphi] = 0\,. \end{aligned} \tag{2.1}$$

Eq. (2.1) corresponds to the most natural commutation relations to impose when a problem with cylindrical symmetry is to be studied. In principle, $\theta(r, t)$ in (2.1) is an arbitrary function. However, noncommutativity in curved space-time imposes severe restrictions on the function $\theta(r, t)$. In general, given a two-dimensional space-time with coordinates x^i, $i = 1, 2$ and commutation relations of the general form

$$[x^i, x^j] = i\theta^{ij}(x)\,, \tag{2.2}$$

the associativity of the product is not guaranteed for an arbitrary function $\theta^{ij}(x)$. One can see however that associativity can be achieved whenever

$$\nabla_k \theta^{ij} = 0\,. \tag{2.3}$$

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The unique solution of these equations for $d = 2$ is given by

$$\theta^{ij} = \theta_0 \frac{\varepsilon^{ij}}{\sqrt{g}} \tag{2.4}$$

with θ_0 being a constant.

The two-dimensional curved space-time metric in which the original 4-dimensional Yang-Mills action reduces to an abelian Higgs action turns out to be

$$g^{ij} = r^2 \delta^{ij} \, . \tag{2.5}$$

Of course to exploit this connection one necessarily has to interpret r as a dimensionless variable. This can be achieved by starting from dimensionless variables in euclidean four dimensional space (through the introduction of a length scale R which can be related with the instanton size). Alternatively one can introduce a dimensionful noncommutative parameter $\theta = R^2\theta_0$.

Then, using solution (2.4), we see that the commutation relations (2.1) to impose should take the form

$$[r,t] = ir^2\theta_0 \, ; \qquad \text{all other} \quad [.,.] = 0 \tag{2.6}$$

with r and t dimensionless variables in the two-dimensional curved space. The connection with dimensionful variables r' and t' goes as follows. The two-dimensional curved metric (2.5) should be written in the form

$$g^{ij} = \frac{r'^2}{R^2}\delta^{ij} \tag{2.7}$$

so that the commutation rule (2.6) becomes

$$[r', t'] = ir'^2\theta_0 \, . \tag{2.8}$$

One can easily show that this commutation rule coincides with that studied in [28]. From here on we shall work with dimensionless variables and recover the scale at the end of the calculations.

A simplification occurs after the observation that

$$r * t - t * r = ir^2\theta_0 \qquad \Rightarrow \qquad t * \frac{1}{r} - \frac{1}{r} * t = i\theta_0 \, . \tag{2.9}$$

Then, introducing $y^1 = -1/r$ and $y^2 = t$ eq. (2.6) becomes a usual two-dimensional Moyal product,

$$[y^1, y^2] = i\theta_0 \, . \tag{2.10}$$

Axially symmetric multi-instanton solutions to the selfduality equations

$$F_{\mu\nu} = \pm \tilde{F}_{\mu\nu} \tag{2.11}$$

were found in [36] by making a noncommutative extension (with U(2) gauge group) of the cylindrically symmetric ansatz considered by Witten [27]. For the SU(2) sector one just proposes the same ansatz as in ordinary space,

$$
\begin{aligned}
\vec{A}_1 &= A_1(y^1,y^2)\vec{\Omega}(\vartheta,\varphi)\\
\vec{A}_2 &= A_2(y^1,y^2)\vec{\Omega}(\vartheta,\varphi) && (2.12)\\
\vec{A}_\vartheta &= \phi^1(y^1,y^2)\partial_\vartheta\vec{\Omega}(\vartheta,\varphi) + \left(1+\phi^2(y^1,y^2)\right)\vec{\Omega}(\vartheta,\varphi)\wedge\partial_\vartheta\vec{\Omega}(\vartheta,\varphi)\\
\vec{A}_\varphi &= \phi^1(y^1,y^2)\partial_\varphi\vec{\Omega}(\vartheta,\varphi) + \left(1+\phi^2(y^1,y^2)\right)\vec{\Omega}(\vartheta,\varphi)\wedge\partial_\varphi\vec{\Omega}(\vartheta,\varphi) && (2.13)
\end{aligned}
$$

with

$$
\vec{\Omega}(\vartheta,\varphi) = \begin{pmatrix}\sin\vartheta\cos\varphi\\ \sin\vartheta\sin\varphi\\ \cos\vartheta\end{pmatrix}. \tag{2.14}
$$

Concerning the remaining U(1) components, it is natural to propose the ansatz

$$
\begin{aligned}
A_1^4 &= A_1^4(y^1,y^2)\\
A_2^4 &= A_2^4(y^1,y^2)\\
A_\vartheta^4 = A_\varphi^4 &= 0\,.
\end{aligned} \tag{2.15}
$$

With this ansatz, the selfduality equations (2.11) become

$$
\begin{aligned}
\partial_2A_1-\partial_1A_2+\frac{i}{2}\left[A_2,A_1^4\right]+\frac{i}{2}\left[A_2^4,A_1\right] &= 1-\left(\phi^1\right)^2-\left(\phi^2\right)^2\\
\partial_2A_1^4-\partial_1A_2^4+\frac{i}{2}\left[A_2^4,A_1^4\right]+\frac{i}{2}\left[A_2,A_1\right] &= -i\left[\phi^1,\phi^2\right]\\
\partial_2\phi^1+\frac{1}{2}\left[A_2,\phi^2\right]_++\frac{i}{2}\left[A_2^4,\phi^1\right] &= \left(y^1\right)^2\left(\partial_1\phi^2-\frac{1}{2}\left[A_1,\phi^1\right]_++\frac{i}{2}\left[A_1^4,\phi^2\right]\right)\\
\partial_2\phi^2-\frac{1}{2}\left[A_2,\phi^1\right]_++\frac{i}{2}\left[A_2^4,\phi^2\right] &= -\left(y^1\right)^2\left(\partial_1\phi^1+\frac{1}{2}\left[A_1,\phi^2\right]_++\frac{i}{2}\left[A_1^4,\phi^1\right]\right).
\end{aligned} \tag{2.16}
$$

Imposing the further restriction in the U(1) sector,

$$
\begin{aligned}
A_t^4(u,t) &= A_t(u,t)\\
A_u^4(u,t) &= A_u(u,t)
\end{aligned} \tag{2.17}
$$

and introducing the notation

$$
\begin{aligned}
\phi &= \phi^1 - i\phi^2\\
D\phi &= \partial\phi + iA\phi\\
F_{12} &= \partial_1A_2-\partial_2A_1+i\left[A_1,A_2\right]
\end{aligned} \tag{2.18}
$$

the system (2.16) reduces to

$$
F_{12} = \frac{1}{2}[\phi,\bar{\phi}] \tag{2.19}
$$

$$
F_{12} = \frac{1}{2}[\phi,\bar{\phi}]_+ - 1 \tag{2.20}
$$

$$
D_2\phi = i\left(y^1\right)^2 D_1\phi\,. \tag{2.21}
$$

Although this system is overconstrained, nontrivial solutions were obtained in [36] within the Fock space framework. In this approach, the noncommutative coordinates algebra defined by (2.10) is viewed as an algebra of annihilation and creation operators,

$$\begin{aligned} a &= \frac{1}{\sqrt{2\theta_0}}\left(y^1 + iy^2\right) , \qquad a^\dagger = \frac{1}{\sqrt{2\theta_0}}\left(y^1 - iy^2\right) \\ \left[a, a^\dagger\right] &= 1 \,. \end{aligned} \tag{2.22}$$

Given a field χ, one associates an operator O_χ acting on Fock space as

$$O_\chi(a, a^\dagger) = \frac{1}{4\pi^2\theta_0}\int d^2k \tilde{\chi}(k, \bar{k}) \exp\left(-i\left(\bar{k}a + ka^\dagger\right)\right) . \tag{2.23}$$

The star product of fields in configuration space becomes just the operator product in Fock space.

$$O_\eta O_\chi = O_{\eta * \chi} \,. \tag{2.24}$$

Here the Moyal $*$-product of two functions η and χ is defined as

$$\eta(x) * \chi(x) = \exp\left(\frac{i}{2}\theta^{ij}\partial_i^x\partial_j^y\right)\eta(x)\chi(y)\Big|_{y=x} \,. \tag{2.25}$$

Derivatives in configuration space should be replaced by commutators in Fock space,

$$\partial_z \to -\frac{1}{\sqrt{\theta_0}}[a^\dagger, \ ] , \qquad \partial_{\bar{z}} \to \frac{1}{\sqrt{\theta_0}}[a, \ ] , \tag{2.26}$$

where we have written

$$z = \frac{1}{\sqrt{2}}\left(y^1 + iy^2\right) . \tag{2.27}$$

Now, compatibility of equations (2.19) and (2.20) implies

$$\bar{\phi}\phi = 1 , \qquad \phi\bar{\phi} = 1 + 2F_{12} \tag{2.28}$$

and hence a nontrivial solution exists in the form of a shift operator,

$$\phi = \sum_{n=0} |n + q\rangle\langle n| \,. \tag{2.29}$$

Here $\{|n\rangle\}$ is the Fock space basis of eigenfunctions of the number operator $N = a^\dagger a$ and the integer $q \geq 0$ is related to the topological charge. Now, consistency of this last equation with eq. (2.21) completely fixes A_z,

$$\begin{aligned} A_z = &-\frac{i}{\sqrt{\theta_0}}\sum_{n=0}^{q-1}\left(\sqrt{n+1}\right)|n+1\rangle\langle n| + \\ &+ \frac{i}{\sqrt{\theta_0}}\sum_{n=q}\left(\sqrt{n+1-q} - \sqrt{n+1}\right)|n+1\rangle\langle n| \end{aligned} \tag{2.30}$$

provided that

$$\theta_0 = 2 \,. \tag{2.31}$$

In particular, both the l.h.s. and r.h.s. of eq. (2.21) vanish separately. Regarding the particular value of θ_0 for which the solution was found, let us recall that also for vortices in flat space it was necessary to fix θ_0 (but in that case to the value $\theta_0 = 1$), in order to satisfy the corresponding Bogomol'nyi equations.

The magnetic field $B = iF_{z\bar{z}}$ associated with solution (2.30) takes the form,

$$B = -\frac{1}{2}\left(|0\rangle\langle 0| + \cdots + |q-1\rangle\langle q-1|\right) \tag{2.32}$$

with associated magnetic flux

$$\Phi = 2\pi TrB = -\pi q\,. \tag{2.33}$$

A factor $\pi\theta_0$ was included in the definition of the magnetic flux, one half of the usual factor since one is working in the half plane.

Each projector $|n\rangle\langle n|$ in Fock space can be related to a Laguerre polynomial in configuration space through the connection

$$|n\rangle\langle n| \to 2(-1)^n \exp\left(-\frac{(y^1)^2+(y^2)^2}{2}\right) L_n\left((y^1)^2+(y^2)^2\right). \tag{2.34}$$

Then, since the Laguerre polynomial L_n is concentrated in an annulus of radius R_n, growing with n according to $R_n \sim \sqrt{n}$, one can view the magnetic flux (2.32) as that of a superposition of q annular vortices of unit flux. This should be compared with the multi-instanton solution in ordinary space, for which the corresponding q-vortex is a superposition of q 1-vortices centered at arbitrary points along the time axis.

We can now easily write the selfdual multi-instanton solution in 4-dimensional space by inserting the solution (2.29) and (2.30) into the ansatz (2.13). The resulting selfdual field strength reads

$$\vec{F}_{21} = B\vec{\Omega} \tag{2.35}$$
$$\vec{F}_{\vartheta\varphi} = B\sin\vartheta\,\vec{\Omega} \tag{2.36}$$
$$F^4_{21} = B \tag{2.37}$$
$$F^4_{\vartheta\varphi} = B\sin\vartheta \tag{2.38}$$

with the other field-strength components vanishing. The instanton number is given by

$$Q = \frac{1}{32\pi^2}\mathrm{tr}\int d^4x\varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} = \frac{1}{\pi}\int_{-\infty}^{0} dy^1 \int_{-\infty}^{\infty} dy^2 B^2 = 2\,\mathrm{Tr}\,B^2 = \frac{q}{2}\,. \tag{2.39}$$

3. Gauge choices

As stated in the introduction, Manton [35] developed a procedure (that implies taking the limit of infinite topological charge) that effectively reduces the 4 dimensional cylindrically symmetric multi-instanton configuration in ordinary space to a static monopole solution of the Bogomol'nyi-Prasad-Sommerfield equations. In order to extend this procedure to the noncommutative case, we shall need to consider the instanton configuration described in the precedent section in an appropriate gauge ensuring that, after taking the $q \to \infty$

limit, one ends, after an appropriate time-dependent gauge transformation, with a static configuration so that the remaining spatial dependence will be consistent with static BPS equations of a Yang-Mills-Higgs system.

Now, as we shall see, after taking the $q \to \infty$ limit of the noncommutative instanton described above, the gauge field configuration, as it happens in the commutative case, remains time dependent. This is due to the fact that the 2-dimensional vortex solution from which it was constructed, originally in the Lorentz gauge, becomes, in the infinite charge limit, a linear function with one of its components depending on t. In ordinary space, such a linear dependence on time can be easily eliminated by an appropriate gauge transformation but the procedure becomes delicate in the noncommutative case. We shall then discuss this point (at the level of the vortex solution), before proceeding to the analysis of the resulting BPS equations.

Let us consider a $\mathrm{U}_*(1)$ linear gauge potential in $d = 2$ dimensions, in the Lorentz gauge,

$$\mathcal{A}_i = \frac{\mathcal{B}}{2}\varepsilon_{ij}x^j\,, \qquad i,j = 1,2\,, \tag{3.1}$$

where the commutation relations for coordinates are

$$[x^1, x^2] = i\theta_0\,. \tag{3.2}$$

The field strength takes the form

$$\begin{aligned} F_{12} &= \partial_1\mathcal{A}_2 - \partial_2\mathcal{A}_1 + i\left(\mathcal{A}_1 * \mathcal{A}_2 - \mathcal{A}_2 * \mathcal{A}_1\right) \\ &= -\mathcal{B} - \frac{\mathcal{B}^2\theta_0}{4}\,. \end{aligned} \tag{3.3}$$

The first term in the second line of (3.3) is just the field strength that would arise in the commutative case, while the second is due to the fact we are dealing with the noncommutative U(1) gauge group, which we denote by $\mathrm{U}_*(1)$. With our conventions, the covariant derivative in the adjoint reads

$$D_i = \partial_i + i[\mathcal{A}_i,\ ]\,. \tag{3.4}$$

Considering a gauge transformation under which gauge fields change as

$$\mathcal{A}'_i = g^{-1} * \mathcal{A}_i * g - ig^{-1} * \partial_i g \tag{3.5}$$

$$F'_{ij} = g^{-1} * F_{ij} * g \tag{3.6}$$

then, eq. (3.5) can be written in the form

$$\mathcal{A}'_i = \mathcal{A}_i + g^{-1} * [\mathcal{A}_i, g] - ig^{-1} * \partial_i g\,. \tag{3.7}$$

Now, in view of the explicit form of the gauge field configuration (3.1) one has

$$[\mathcal{A}_i, g] = -i\frac{\mathcal{B}\theta_0}{2}\partial_i g \tag{3.8}$$

so that, finally, eq. (3.7) becomes

$$\mathcal{A}'_i = \mathcal{A}_i - i\alpha g^{-1}\partial_i g\,, \tag{3.9}$$

where

$$\alpha = 1 + \frac{\mathcal{B}\theta_0}{2}\,. \tag{3.10}$$

We then see that, if one only allows for regular gauge transformations, the gauge orbit to which $\mathcal{A}_i$ belongs consists, for $\mathcal{B}\theta_0 = -2$, of just one point. But it is precisely the value to which our multi-vortex solution tends in the $q \to \infty$ limit. As we shall show by allowing gauge transformations singular at $\mathcal{B}\theta_0 = -2$, one is able to gauge away the $\mathcal{A}_1$ component of the configuration (3.1).

In the commutative case, one easily finds that the transformation corresponds to the gauge group element

$$g_c = \exp\left(-i\frac{\mathcal{B}}{2}x^1x^2\right). \tag{3.11}$$

We then propose the following ansatz for the gauge transformation in the noncommutative case,

$$g_{nc} = A\exp\left(-i\beta x^1x^2\right), \tag{3.12}$$

where

$$\beta = \frac{\mathcal{B}}{1+\alpha} \tag{3.13}$$

and A is an arbitrary parameter to be appropriately adjusted. Note that the exponential in (3.12) is defined with the ordinary product in its series expansion

$$g_{nc} = A\left(1 - i\left(\beta x^1x^2\right) - \frac{1}{2!}\left(\beta x^1x^2\right)\left(\beta x^1x^2\right) + \cdots\right). \tag{3.14}$$

Because of this fact, it is not a priory guaranteed that g_{nc} is a unitary element of the noncommutative gauge group $\mathrm{U}_*(1)$. We shall see however that one can chose A so that $g_{nc} \in \mathrm{U}_*(1)$. To see this, it will be convenient to use the Weyl-Moyal connection (2.23),

$$\hat{g}_{nc}(\hat{x}_1,\hat{x}_2) = \int \frac{d^2p}{(2\pi)^2}\,\tilde{g}_{nc}(p)\;e^{i(p_1\hat{x}_1+p_2\hat{x}_2)}\,, \tag{3.15}$$

where $\hat{x}_1$ and $\hat{x}_2$ are operators satisfying the noncommutative algebra,

$$[\hat{x}_1,\hat{x}_2] = i\theta\,. \tag{3.16}$$

In this framework, the product of operators can be written in Fourier space as

$$\hat{f}(\hat{x}_1,\hat{x}_2)\cdot\hat{h}(\hat{x}_1,\hat{x}_2) \to \int \frac{d^2q}{(2\pi)^2}\,\tilde{f}(p-q)\tilde{h}(q)\;\exp(i(p_1q_2-p_2q_1)\theta_0)\,. \tag{3.17}$$

For the ansatz (3.12) one has

$$\tilde{g}_{nc}(p_1,p_2) = \frac{2\pi A}{\beta}\exp\left(i\frac{p_1p_2}{\beta}\right). \tag{3.18}$$

Then, after some straightforward calculation, one finds

$$\hat{g}_{nc}(\hat{x}_1,\hat{x}_2)\cdot\hat{g}_{nc}(\hat{x}_1,\hat{x}_2)^\dagger = \frac{|A|^2}{1-(\theta_0\beta/2)^2}\,.$$

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Finally, with an appropriate choice for A one can write the unitary gauge transformation $g_{nc} \in \mathrm{U}_*(1)$ in the form

$$g_{nc} = \frac{\sqrt{1+\mathcal{B}\theta_0/2}}{1+B\theta_0/4}\exp\left(-i\frac{\mathcal{B}}{2(1+\mathcal{B}\theta_0/4)}x_1x_2\right). \tag{3.19}$$

Under this gauge transformation, which as expected is singular at $\theta_0\mathcal{B} = -2$, one manages to gauge out the $\mathcal{A}_1$ component in (3.1),

$$\mathcal{A}_1' = g_{nc}^{-1} * \mathcal{A}_1 * g_{nc} - ig_{nc}^{-1} * \partial_i g_{nc} = 0\,. \tag{3.20}$$

Let us now uplift this transformation to the full gauge group $\mathrm{U}_*(2)$, in order to eliminate an A_1 linear component in the original 4 dimensional ansatz (2.13) and (2.15). We propose the following gauge group transformation

$$g_{\mathrm{U}(2)} = \exp_*\left(-ic[y^1,y^2]_+\Lambda\right) \tag{3.21}$$

with $[y^1,y^2]_+$ the Moyal anticommutator of y^1 and y^2 and

$$c = \frac{1}{2\theta_0}\log\left(1+\frac{\mathcal{B}\theta_0}{2}\right) \tag{3.22}$$

$$\Lambda = \frac{1}{2}(\Omega^a\sigma^a + I)\,. \tag{3.23}$$

The notation $\exp_*$ means that this exponential is defined using the Moyal product in its series expansion.

One can easily see that

$$g_{\mathrm{U}(2)}^\dagger = g_{\mathrm{U}(2)}^{-1} = 1 + \Lambda\left(g_{nc}^\dagger - 1\right). \tag{3.24}$$

The $\mathrm{U}_*(2)$ gauge transformation for the $i = 1,2$ components of the A_i transform according to

$$\begin{aligned} A_i' &= g_{\mathrm{U}(2)}^{-1} * A_i * g_{\mathrm{U}(2)} + ig_{\mathrm{U}(2)}^{-1} * \partial_i g_{\mathrm{U}(2)} \\ &= \Lambda\left(g_{nc}^{-1} * A_i * g_{nc} + ig_{nc}^{-1} * \partial_i g_{nc}\right), \qquad i = 1,2 \end{aligned} \tag{3.25}$$

so, in view of (3.20), one can gauge out the linear time dependent component A_1 of the gauge field configuration leading to the field strength (2.38).

4. Monopoles from instantons

Let us now consider the limit of infinite topological charge in order to construct static, spherically symmetric BPS solutions from axially symmetric ones. First, taking the $q \to \infty$ limit in eq. (2.32) one gets a constant magnetic field,

$$\lim_{q\to\infty} B = -\frac{1}{2}\sum_{n=0}^{\infty}|n\rangle\langle n| = -\frac{1}{2}\,. \tag{4.1}$$

Such a magnetic field follows from the gauge field configuration (see eq. (2.30))

$$\lim_{q\to\infty} A_z = -\frac{i}{\sqrt{2}}\sum_{n=0}^{\infty}\left(\sqrt{n+1}\right)|n+1\rangle\langle n| = -\frac{i}{\sqrt{2}}\sum_{n=0}^{\infty} a^\dagger |n\rangle\langle n| = -\frac{i}{2}\bar{z}\,. \tag{4.2}$$

Recalling that $A_z = (1/\sqrt{2})(A_1 - iA_2)$ we have

$$\lim_{q\to\infty} A_1 = -\frac{y^2}{2}\,, \qquad \lim_{q\to\infty} A_2 = \frac{y^1}{2}\,. \tag{4.3}$$

In order to convert the instanton selfduality equations (2.11) into static BPS equations for a Yang-Mills-Higgs system, one first needs to identify the time component A_2 of the gauge field with a Higgs scalar Φ taking values in the Lie algebra of $\mathrm{U}_*(2)$. The spatial components $(A_1, A_\vartheta, A_\varphi)$ will be identified with the spatial components of a Yang-Mills field that we shall denote B_i. That is, taking $B_0 = 0$ one establishes the following connection

$$\begin{aligned} A_2 &\to \Phi \\ (A_r, A_\vartheta, A_\varphi) &\to (B_r, B_\vartheta, B_\varphi) \\ F_{ij} &\to G_{ij} = \partial_i B_j - \partial_j B_i + i[B_i, B_j]\,. \end{aligned} \tag{4.4}$$

Now, in order to obtain a noncommutative $\mathrm{U}_*(2)$ monopole like static solution (B_i, Φ) from the instanton solution A_μ as defined in (2.12) and (2.13) one needs a time-independent field configuration. While the $q \to \infty$ limit does lead to a static configuration for the Higgs field Φ, this is not the case for the gauge field components. The A_1 component exhibits a linear dependence on $y^2 = t$, as given by eq. (4.3), which could be gauged away, but subject to a proviso related to the discussion in section 3. Indeed, we have seen that a two-dimensional configuration of the type (3.1), with $\mathcal{B} = -1$ (or $B\theta_0 = -2$) exhibits a gauge orbit consisting of just one point and the same happens for our 4-dimensional $\mathrm{U}_*(2)$ configuration. Then, to gauge away the y^2 (time) dependence of A_1 we are forced to consider singular gauge transformations of the kind discussed in section 3. Indeed, under a gauge transformation of the form (3.21)

$$g_{\mathrm{U}(2)} = \exp_*\left(-ic[y^1, y^2]_+ \Lambda\right) \tag{4.5}$$

A_1 vanishes while A_2 becomes

$$A_2 = -\mathcal{B}\left(1 + \frac{\mathcal{B}\theta_0}{4}\right) x^1 \Lambda = \frac{1}{2} x^1 \Lambda\,. \tag{4.6}$$

Then, the $\mathrm{U}_*(2)$ Higgs scalar $\Phi = A_2$ is just

$$\Phi = \frac{1}{2} x^1 \Lambda = -\frac{1}{2r}\Lambda\,. \tag{4.7}$$

Finding the actual gauge transformation that eliminates the time dependence from the angular components is far more complicated. However, we know that the in the $q \to \infty$ limit the only non-trivial strength components of the gauge field, as given by eqs. (2.35)–(2.38) take the very simple form

$$\begin{aligned} \vec{F}_{0r} &= \frac{B}{r^2}\,\vec{\Omega}\,, \qquad & \vec{F}_{\vartheta\varphi} &= B \sin\vartheta\, \vec{\Omega} \\ F^4_{0r} &= \frac{B}{r^2}\,, \qquad & F^4_{\vartheta\varphi} &= B \sin\vartheta \end{aligned} \tag{4.8}$$

with $B = -1/2$. One can then easily find a time-independent instanton configuration leading to such a field strength. It is simply given by

$$\begin{aligned}
\vec{A}'_0 &= \frac{B}{r}\,\vec{\Omega}\,, \qquad \vec{A}'_r = 0\,, \qquad \vec{A}'_\vartheta = -\vec{\Omega}\wedge\partial_\vartheta\vec{\Omega}\,,\\
\vec{A}'_\varphi &= -\vec{\Omega}\wedge\partial_\varphi\vec{\Omega} - (B+1)(1+\cos\vartheta)\vec{\Omega}\\
{A'}^4_0 &= \frac{B}{r}\,, \qquad {A'}^4_r = 0\,, \qquad {A'}^4_\vartheta = 0\,, \qquad {A'}^4_\varphi = -B(1+\cos\vartheta)\,.
\end{aligned} \tag{4.9}$$

Since for nonabelian gauge theories the field strength does not determine the gauge potential up to gauge transformations, as was shown by Wu and Yang in his classic article [31], is not obvious that the fields A'_μ in (4.9) are gauge equivalent to the original instanton configuration A_μ. However we will show that this is in fact the case, the gauge configurations A'_μ and A_μ are related by a gauge transformation.

To see this we notice that both gauge configurations generate the same field strength and satisfy the same equations of motion. Concerning the Bianchi identities, they are both satisfied everywhere except at the origin where they both have the same delta function singularity (see the discussion below). Most of the components of $F_{\mu\nu}$ vanishes, so that from the equation of motion we deduce the following identities

$$\begin{aligned}
D_0 F_{0r} &= 0\,, \qquad D_r F_{0r} = -\frac{2}{r}\,F_{0r}\\
D_\vartheta F_{\vartheta\varphi} &= 0\,, \qquad D_\varphi F_{\vartheta\varphi} = 0
\end{aligned} \tag{4.10}$$

and from the Bianchi identities

$$\begin{aligned}
D_\vartheta F_{0r} &= 0\,, \qquad D_\varphi F_{0r} = 0\\
D_0 F_{\vartheta\varphi} &= 0\,, \qquad D_r F_{\vartheta\varphi} = 2\pi\delta^{(3)}\Lambda\,.
\end{aligned} \tag{4.11}$$

Then we see that all the covariant derivatives of $F_{\mu\nu}$ vanishes, except for $D_r F_{0r} = -\frac{2}{r}\,F_{0r}$ and for that in (4.11) having a delta function singularity. And since $A_r = A'_r = 0$, we conclude that all higher covariant derivatives of the field strength coincide for both configurations. This is precisely the condition ensuring that there exist a gauge transformation connecting A_μ and A'_μ [32]–[33]. So that we conclude that (4.9) is gauge-equivalent to the original gauge field configuration one gets in the $q\to\infty$ limit.

Then, we can write the resulting BPS equation for the $\mathrm{U}_*(2)$ Yang-Mills-Higgs system and its monopole solution in the form

$$\frac{1}{2}\varepsilon^{ijk}G_{jk} = D^i\Phi \tag{4.12}$$

$$\vec{\Phi} = -\frac{1}{2r}\,\vec{\Omega}\,, \qquad \vec{B}_r = 0\,, \qquad \vec{B}_\vartheta = -\vec{\Omega}\wedge\partial_\vartheta\vec{\Omega}\,, \tag{4.13}$$

$$\vec{B}_\varphi = -\vec{\Omega}\wedge\partial_\varphi\vec{\Omega} + \frac{1}{2}(1+\cos\vartheta)\vec{\Omega} \tag{4.14}$$

$$\Phi^4 = -\frac{1}{2r}\,, \qquad B^4_r = 0\,, \qquad B^4_\vartheta = 0\,, \qquad B^4_\varphi = -\frac{1}{2}(1+\cos\vartheta)\,. \tag{4.15}$$

With this time-independent configuration we can make the correspondence (4.4) and obtain a BPS monopole. Note that both the SU(2) and U(1) components of B_φ have a contribution $1/2(1+\cos\vartheta)$ which coincide with the Wu-Yang and Dirac singular monopole configuration. In order to compute the corresponding magnetic charge, we define, as usual, an "electromagnetic" field strength $\mathcal{G}_{ij}$ by projecting the $\mathrm{U}_*(2)$ field strength along the Φ direction,

$$\mathcal{G}_{ij} = \mathrm{tr}\left(\frac{\Phi}{|\Phi|}G_{ij}\right) \tag{4.16}$$

which leads to a magnetic field of the form

$$B^r = -\frac{1}{r^2} \tag{4.17}$$

corresponding to a unit charge magnetic monopole

$$Q_m = \frac{1}{4\pi}\Phi_m = -1 \tag{4.18}$$

with Φ_m the magnetic flux associated to (4.17). The corresponding electric field, consistently defined as

$$\mathcal{G}_{i0} = \mathrm{tr}\left(\frac{\Phi}{|\Phi|}G_{i0}\right) \tag{4.19}$$

of course vanishes. So, we have arrived to a magnetic monopole-like solution of first order (BPS) equations

$$D_i\Phi = \frac{1}{2}\varepsilon_{ijk}G_{jk} \tag{4.20}$$

which are those giving the extrema for the energy of a gauge field-Higgs system. Then, apart from the fact that there is a Dirac-Wu-Yang singularity, the configuration solves the second order Yang-Mills-Higgs equations of motion,

$$\begin{aligned} D_i G^{ij} &= [\Phi, D^j\Phi] \\ D_i D^i \Phi &= 0\,. \end{aligned} \tag{4.21}$$

Of course, the energy associated to the solution (4.15),

$$E = \mathrm{Tr}\int d^3x\left(D_i\Phi D_i\Phi + \frac{1}{2}F_{ij}F_{ij}\right) \tag{4.22}$$

is strictly infinite (as it coincides with the selfenergy of a Dirac monopole)

$$E = \pi\int dr\frac{1}{r^2} = \int d^3x B^2_{\mathrm{mon}}\,. \tag{4.23}$$

Now, if we introduce a regulator ϵ[1] to cut off the short-distances divergence and recover the dimensional scale R ($\theta = \theta_0 R^2 = 2R^2$) we can write E in the form

$$E = \frac{\pi}{g^2_{\mathrm{YM}}R\epsilon} = \frac{\pi R}{g^2_{\mathrm{YM}}R^2\epsilon} = \frac{2\pi}{g^2_{\mathrm{YM}}\theta}\frac{R}{\epsilon}\,. \tag{4.24}$$

[1]Regulator ϵ is dimensionless since r is a dimensionless variable.

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(We have reintroduced the gauge coupling constant $g_{\rm YM}$ which was taken equal to 1 along the paper). Defining a length $L = R/\epsilon$ we see that E can be identified with the mass of a string of length L whose tension is

$$T = \frac{2\pi}{g_{\rm YM}^2\theta}\,. \tag{4.25}$$

One can see (4.22) as emerging in the decoupling linearized limit of a $D3$-brane in the Type IIB string theory with the Higgs field describing its fluctuations in a transverse direction.[2] Since the B-field leading to our noncommutative setting is transverse to the $D3$-brane surface, one can make an analysis similar to that presented by Callan-Maldacena in [34] with the scalar field describing a perpendicular spike. In this last investigation, where the electric case is discussed, the string interpretation corresponds to an F-string attached to a $D3$-brane. Our magnetic case can be related to this by an S-duality transformation changing the $F1$ into a $D1$ string. Comparing the tension of such a $D1$-string with the one resulting from our solution (eq. (4.25)),

$$T_{D1} = \frac{1}{2\pi\alpha' g_s} = \frac{2\pi}{g_{\rm YM}^2\theta} \tag{4.26}$$

and using $2\pi g_s = g_{\rm YM}^2$ we see that quantization of the magnetic monopole charge leads to a quantized value for θ in string length units equal to 1 for our charge-1 monopole, $\theta/2\pi\alpha' = 1$.

5. Discussion

We shall summarize here our results and discuss the properties of the noncommutative monopole solution we have found as compared with previous constructions.

Previous investigations on noncommutative monopoles [22]–[24] were based in Nahm's construction in ordinary space [29]. These works start from the ADHM version of the noncommutative multi-instanton and for the $\mathrm{U}_*(1)$ gauge group, lead to a BPS solution which has zero magnetic charge.

The alternative route we have taken, parallels in noncommutative space, the observation of refs. [35], by taking the infinite charge limit of an axially symmetric (in time) instanton. The resulting configuration solves the BPS equations for a Yang-Mills-Higgs system with the original A_0 gauge field component playing the role of the scalar field.

In both approaches -that of ref. [22]–[24] and ours- one needs to start from a multi-instanton configuration in noncommutative 4-dimensional space. If one follows the Nahm approach, one needs a noncommutative version of the ADHM solution and this was presented in [3]. The noncommutative solution corresponds to a self-dual $\theta^{\mu\nu}$ which means that the noncommutative relations are reduced to the nontrivial pair $[x^1,x^2] = [x^3,x^4] = i\theta$. In contrast, the axially symmetric instanton solution corresponds to a noncommutative relation of the form $[r,t] = i\theta(r,t)$ [36] (Covariance arguments force the condition $\theta(r,t) = r^2\theta_0$).

[2] We thank the referee for clarifying to us the correct brane interpretation of the solution.

When the 4-dimensional original problem is reduced to three dimensions, these different commutation relations lead, of course, to different noncommutative spaces. In particular, one could think that in our construction, for which noncommutativity necessarily involves time, static configurations could just be considered as ordinary commutative ones. However this configuration has a genuine noncommutative origin as a descendent of the noncommutative instanton (2.35)–(2.38). Moreover since solitons are intended to play a role through nonperturbative effects where all space-time variables come into play, their noncommutative character manifests, as it happens for example when one computes tension (4.25) from the string-monopole mass formula.

It is worthwhile to emphasize how well Manton's method works for the noncommutative instanton (2.35)–(2.38) leading, as in ordinary space, to a time-independent configuration satisfying the BPS equations. And also how different are the final products: a 't Hooft-Polyakov monopole in ordinary space and a Wu-Yang monopole in the present case.

An application to brane dynamics of noncommutative monopoles was given in [24] for the case a static BPS $U_*(1)$ solution obtained from an ADHM instanton. Now, the soliton obtained from the ADHM noncommutative instanton has zero magnetic charge, a result that can be understood in terms of a system of a magnetic monopole attached to a flux tube of opposite charge, transverse to the noncommutative plane. In contrast, we have shown that the charge of the solution we obtained is effectively 1. Studying the second order equations of motion associated to our BPS solution, we have seen that our soliton corresponds to a Wu-Yang singular configuration: although it verifies exactly the BPS first order equations, delta-function sources are needed in the second order Euler-Lagrange equations.

Let us finally point a direction along which it would be worthwhile to pursue our investigation. As already mentioned, the reduction from selfdual to BPS equations could be performed with the isometry k_μ not necessarily in the euclidean time direction. In particular, a different choice for k^μ leads in ordinary space to monopoles on H^3, hyperbolic 3-spaces, as defined in [30]. Instead of the noncommutative axially symmetric (with axis in time) instantons we started from, one should consider axially symmetric invariant noncommutative instantons but in this case with "axis" in $\mathbb{R}^2 \sim S^1 \subset \mathbb{R}^4$. The properties of the resulting monopoles in the corresponding noncommutative space will change drastically and can exhibit interesting features. We hope to come back to this problem in the future.

Acknowledgments

We wish to thank Carlos Nuñez for helpful comments. This work was partially supported by UNLP, CICBA, CONICET, ANPCYT (PICT grant 03-05179) Argentina and ECOS-Sud Argentina-France collaboration (grant A01E02). D.H.C . was partially supported by Fundación Antorchas.

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